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Generalized $SU_q(1|2)$ Gaudin model

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Abstract

In this paper we propose the Hamiltonians of the generalized $SU_q(1|2)$ Gaudin model corresponding to the periodic generalized t–J model. With the help of the well defined graded quantum determinant, we obtain the eigenstate and eigenvalues of the generating function and the Hamiltonians of the Gaudin model in the fermionic background in the framework of the graded quantum inverse scattering method. The Bethe ansatz equations are also obtained.

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1. Introduction

The Gaudin model associated with the $SU(2)$ Lie algebra was first constructed by Gaudin [1–3] in 1973. By using the off-shell Bethe ansatz method, Babujian and Flume [4] generalized it into the general Lie algebra g case, and constructed the corresponding Hamiltonian without the solution. In principle, one can apply Gaudin's method to find the solution, however, this is highly tedious. In order to solve this problem, Feigin *et al* [5] proposed a new method and obtained the solution and the Bethe ansatz equations. The Gaudin-like system is a new kind of integrable quantum model with long-range interaction and admits a classical r-matrix structure. So, we can simply consider the Gaudin model as a proper limit of some integrable quantum chains in the framework of the quantum inverse scattering method (QISM) [6, 7]. Sklyanin [8] suggested that the spectrum and eigenfunctions of the spin-$\frac{1}{2}$ Gaudin models with rational and trigonometric interaction could be derived from XXX and XXZ chains. Then, in [9–11], Sklyanin and Takebe obtained the arbitrary-spin XYZ Gaudin model as a quasi-classical limit of the inhomogeneous higher-spin generalization of the XYZ model. In their method, the quantum determinant has played a very important role.

There is a belief that strongly correlated electron systems are important in studying high-temperature superconductivity [12, 13]. An appropriate model is the t–J model suggested by Anderson *et al* [14, 15]. The Hamiltonian includes the nearest-neighbour hopping (t) and anti-ferromagnetic exchange (J). In one dimension, a generalized t–J model reads [16]

$$H = -t \sum_{j=1}^{N-1} \sum_{\sigma} (C_{j\sigma}^{\dagger} C_{j+1\sigma} + C_{j+1\sigma}^{\dagger} C_{j\sigma})$$

$$-J\sum_{j=1}^{N-1}[S_j^xS_{j+1}^x+S_j^yS_{j+1}^y+\cos(\eta)(S_j^zS_{j+1}^z-\tfrac{1}{4}n_jn_{j+1})]$$
$$-t\cos(\eta)\sum_{j=1}^{N}n_j+\mathrm{i}t\sin(\eta)(n_1-n_N)$$
$$-\mathrm{i}t\sin(\eta)\sum_{j=1}^{N-1}(n_jS_{j+1}^z-S_j^zn_{j+1}) \tag{1}$$

where $C_{j\sigma}^\dagger$ ($C_{j\sigma}$) is the creation (annihilation) operator with spin $\sigma=\uparrow$ or $\downarrow$ on the jth side, n_j is the number of the electron and $\vec{S}_j$ is the spin-$\frac{1}{2}$ operator. The Fock space is spanned by three kinds of vector, $|0\rangle_j$, $|\uparrow\rangle_j$ and $|\downarrow\rangle_j$ representing hole, spin-up and spin-down states respectively, $C_{j,\uparrow}^\dagger|0\rangle_j=|\uparrow\rangle_j$, $C_{j,\downarrow}^\dagger|0\rangle_j=|\downarrow\rangle_j$ and $C_{j,\sigma}|0\rangle_j=0$. The total vacuum can be represented by $\prod_{j=1}^{\otimes N}|0\rangle_j$. η is an anisotropic parameter. It is shown that under the proper condition $J=2t$ the model has a new symmetry—super-symmetric $SU_q(1|2)$, a graded deformed group. When $J=2t$ and $\eta=0$, the model reduces to the usual super-symmetric t–J model.

Essler and Korepin [17] show that the one-dimensional t–J model can be obtained from the transfer matrix of the two-dimensional super-symmetric exactly solvable lattice model. Using the graded QISM [18, 19], they obtain the eigenvalue and eigenstate for the super-symmetric t–J model with periodic boundary conditions in three different backgrounds. The Hamiltonian (1) with periodic boundary was studied by Yue and Qiu [20].

In this paper, we apply Sklyanin's method [8] to a more complicated graded case and obtain the Hamiltonians of the generalized $SU_q(1|2)$ Gaudin model with periodic boundary conditions. Starting from the graded monodromy matrix, we define the graded quantum determinant which commutes with the monodromy matrix. Then, we take the quasi-classical limit ($\eta\to0$) of both the quantum determinant and the monodromy matrix of the $SU_q(1|2)$ t–J model and obtain the Hamiltonians of the generalized $SU_q(1|2)$ Gaudin model. Finally, using the graded QISM, we obtain the eigenstate and the eigenvalues of the generating function and the Hamiltonians of the generalized $SU_q(1|2)$ Gaudin model. It is necessary to point out that the quasi-classical limit procedure is quite subtle. One should be careful to obtain required quantities under such a limit, especially the Hamiltonians and the eigenstate.

This paper is organized as follows. Section 2 gives a description of the generalized t–J model. In section 3, we show how to construct the generalized $SU_q(1|2)$ Gaudin model in detail. The Hamiltonians and the generating function are explicitly obtained. By using the graded QISM method, we also find the eigenstate and eigenvalues. Section 4 includes a brief summary and some discussion.

2. The generalized *t*–*J* model

We first review briefly some facts on the generalized t–J model and the graded version of the QISM. Under the convention $\epsilon_1=\epsilon_2=1$ and $\epsilon_3=0$, the R-matrix is

$$\hat{\boldsymbol{R}}=\sum_{i=1}^{3}\sinh(\eta+(-1)^{\epsilon_i}u)\boldsymbol{E}_{ii}\otimes\boldsymbol{E}_{ii}+\sum_{i\neq j=1}^{3}\sinh(\eta)\mathrm{e}^{\gamma_{ij}u}\boldsymbol{E}_{ii}\otimes\boldsymbol{E}_{jj}$$
$$+\sum_{i\neq j=1}^{3}\sinh(u)(-1)^{\epsilon_i\epsilon_j}\boldsymbol{E}_{ij}\otimes\boldsymbol{E}_{ji} \tag{2}$$

where $\boldsymbol{E}_{ij}$ is defined as $(\boldsymbol{E}_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and

$$\gamma_{ij} = \begin{cases} 1 & i < j \\ -1 & i > j. \end{cases}$$

The L-operators, $L_n(u)$, are constructed from the R-matrix as

$$L_n(u) = \sinh u - [\sinh u - \sinh(\eta - u)][E_n^{11} + E_n^{22}] \\ -[\sinh u - \sinh(\eta + u)]E_n^{33} + \sum_{i\neq k=1}^{3} (-1)^{\epsilon_i\epsilon_k} \mathrm{e}^{\gamma_{ik}u} \sinh u E_n^{ik} \tag{3}$$

where E_n^{ab} are quantum operators acting in the nth quantum space with $(E_n^{ab})_{\alpha\beta} = \delta_{a\alpha}\delta_{b\beta}$.

It is well known that the L-operators satisfy the graded Yang–Baxter relation (YBR) [21–23]

$$\hat{R}_{12}(u - v)L_n(u) \otimes L_n(v) = L_n(v) \otimes L_n(u)\hat{R}_{12}(u - v) \tag{4}$$

with the tensor product $(F \otimes G)_{ac}^{bd} = F_{ab}G_{cd}(-1)^{\epsilon_c(\epsilon_a+\epsilon_b)}$. The N site monodromy matrix $T(u)$ with the shift of spectral parameters δ_m is defined as the product of the L-operators on all sites

$$T(u) = L_N(u - \delta_N)L_{N-1}(u - \delta_{N-1}) \cdots L_1(u - \delta_1) \tag{5}$$

and satisfies the graded YBR

$$\hat{R}(u - v)T(u) \otimes T(v) = T(v) \otimes T(u)\hat{R}(u - v). \tag{6}$$

As a consequence of the graded YBR, the transfer matrices, $t(u) = \mathrm{str}T(u)$, commute with each other for different spectrum parameters, $[t(u), t(v)] = 0$, which ensures the integrability of the system.

In the graded case, the quantum determinant is defined as

$$\Delta(u) \equiv \mathrm{str}_{123}P_{123}^{-}T_{01}(u)T_{02}(u - \eta)T_{03}(u - 2\eta) \quad [\Delta(u), T(v)] = 0 \tag{7}$$

where P_{123}^{-} is a completely anti-symmetric projector

$$P_{123}^{-} = \tfrac{1}{6}[1 - P_{12} - P_{13} - P_{23} + P_{123} + P_{132}] \tag{8}$$

with $[P_{12}]_{a_1a_2}^{b_1b_2} = \delta_{a_1b_2}\delta_{a_2b_1}(-1)^{\epsilon_{a_1}\epsilon_{a_2}}$.

We introduce the local reference state (boson background) $|\Omega\rangle_n = (0, 0, 1)_n^t$ and the whole reference state $|\Omega\rangle = |\Omega\rangle_1 \otimes |\Omega\rangle_2 \cdots |\Omega\rangle_N$. The monodromy matrix acting on the reference state gives

$$\begin{aligned} &T(u)_i^j|\Omega\rangle = 0 \quad i \neq j < 3 \qquad T(u)_3^i|\Omega\rangle \neq 0 \quad i = 1, 2 \\ &T(u)_1^1|\Omega\rangle = |\Omega\rangle \qquad T(u)_2^2|\Omega\rangle = |\Omega\rangle \\ &T(u)_3^3|\Omega\rangle = \prod_{n=1}^{N} \frac{\sinh(u - \delta_n + \eta)}{\sinh(u - \delta_n)}|\Omega\rangle. \end{aligned} \tag{9}$$

In the next section, the eigenvalue of the quantum determinant is a key in determining the eigenvalue of the Gaudin model. The rest of this section will be dedicated to this goal. Expressing $\Delta(u)$, one may have such terms as $T_i^l T_j^i T_l^j$. Using the graded YBR (6), one may arrange them in a standard order. However, this is very tedious at operator level. Fortunately, what we need is just the value of the quantum determinant acting on the reference state $|\Omega\rangle$.

On this state, the calculation becomes much simpler but still has many terms. Hence, we only give a few formulae as examples (they are invalid as the operator formula):

$$T(u)^3_a T(u-\eta)^a_3|\Omega\rangle = -\frac{e^{2\eta}}{2\cosh\eta} T(u)^3_3 T(u-\eta)^a_a|\Omega\rangle + \frac{e^{2\eta}}{2\cosh\eta} T(u)^a_a T(u-\eta)^3_3|\Omega\rangle \quad (10)$$

$$T(u)^3_a T(u-\eta)^b_b T(u-2\eta)^a_3|\Omega\rangle = -e^{2\eta} T(u)^3_3 T(u-\eta)^b_b T(u-2\eta)^a_a|\Omega\rangle \\ +e^{2\eta} T(u)^a_a T(u-\eta)^3_3 T(u-2\eta)^b_b|\Omega\rangle \qquad a=1,2 \quad b=a,3 \quad (11)$$

$$T(u)^3_a T(u-\eta)^b_b T(u-2\eta)^a_3|\Omega\rangle = -\frac{e^{2\eta}}{2\cosh u} T(u)^3_3 T(u-\eta)^b_b T(u-2\eta)^a_a|\Omega\rangle \\ +\frac{e^{2\eta}}{2\cosh u} T(u)^a_a T(u-\eta)^b_b T(u-2\eta)^3_3|\Omega\rangle \\ a=1 \quad b=2 \qquad \text{or} \qquad a=2 \quad b=1 \quad (12)$$

$$T(u)^3_1 T(u-\eta)^1_2 T(u-2\eta)^2_3|\Omega\rangle \\ = \left\{ \frac{e^{3\eta}}{2\cosh u} T(u)^3_3 T(u-\eta)^2_2 T(u-2\eta)^1_1 + e^{2\eta} T(u)^1_1 T(u-\eta)^3_3 T(u-2\eta)^2_2 \right. \\ -e^{2\eta} T(u)^3_3 T(u-\eta)^1_1 T(u-2\eta)^2_2 \\ \left. -\frac{e^{3\eta}}{2\cosh u} T(u)^1_1 T(u-\eta)^2_2 T(u-2\eta)^3_3 \right\} |\Omega\rangle. \quad (13)$$

So, the eigenvalue of the quantum determinant can be written as

$$\Delta|\Omega\rangle = \left\{ -4 + \left(1 - e^{\eta} - \frac{e^{2\eta}}{6\cosh\eta} + \frac{e^{3\eta}}{6\cosh\eta}\right) \prod_{n=1}^{N} \frac{\sinh(u-\delta_n-\eta)}{\sinh(u-\delta_n-2\eta)} \right. \\ + (1-e^{2\eta}) \prod_{n=1}^{N} \frac{\sinh(u-\delta_n)}{\sinh(u-\delta_n-\eta)} \\ \left. + \left(1 + e^{\eta} + e^{2\eta} + \frac{e^{2\eta}}{6\cosh\eta} - \frac{e^{3\eta}}{6\cosh\eta}\right) \prod_{n=1}^{N} \frac{\sinh(u-\delta_n+\eta)}{\sinh(u-\delta_n)} \right\} |\Omega\rangle \\ = \left\{ -1 + 3\eta \sum_{n=1}^{N} \coth(u-\delta_n) + \frac{3N}{2}\eta^2 \right. \\ \left. + 3\eta^2 \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} \coth(u-\delta_n)\coth(u-\delta_m) + o(\eta^3) \right\} |\Omega\rangle. \quad (14)$$

3. The generalized $SU_q(1|2)$ Gaudin model

The Gaudin magnet introduced in [2] was given by taking the quasi-classical limit $\eta \to 0$ of the transfer matrix tr $T(u)$ for the inhomogeneous spin chain [24]. This fact indicates that the Hamiltonian is written in terms of the solution of the classical YBR. This motivates us to define the generalized $SU_q(1|2)$ Gaudin model through the quasi-classical limit of the generalized t–J model. Our strategy is to consider the quasi-classical limit of the proper quantities in the generalized t–J model, such as the graded quantum determinant and the transfer matrix. Then, we can obtain the Hamiltonians and the generating function of the Gaudin model. For convenience, we change the braided R-matrix $\hat{R}$ to the non-braided R-matrix $\boldsymbol{R} = P_{12}\hat{\boldsymbol{R}}$.

Let us examine first the asymptotic behaviour of the operators in the previous section when η tends to 0. The R-matrix, the L-operator and the monodromy matrix are expanded as

$$R_{12}(u) = 1 + \eta r_{12}(u) \qquad L_n(u) = 1 + \eta \mathcal{L}_n(u)$$
$$T(u) = 1 + \eta \mathcal{T}(u) + \eta^2 \mathcal{T}_2(u) + \mathrm{O}(\eta^3) \qquad \mathcal{T}(u) = \sum_{n=1}^{N} \mathcal{L}_n(u - \delta_n) \tag{15}$$

with

$$\mathcal{L}_n(u) = \frac{1}{\sinh u} \begin{pmatrix} -\cosh u E_n^{11} & -\mathrm{e}^{u} E_n^{21} & \mathrm{e}^{u} E_n^{31} \\ -\mathrm{e}^{-u} E_n^{12} & -\cosh u E_n^{22} & \mathrm{e}^{u} E_n^{32} \\ \mathrm{e}^{-u} E_n^{13} & \mathrm{e}^{-u} E_n^{23} & \cosh u E_n^{33} \end{pmatrix}$$
$$r_{12}(u) = \sum_{i=1}^{3} (-1)^{\epsilon_i} \frac{\cosh u}{\sinh u} \boldsymbol{E}_{ii} \otimes \boldsymbol{E}_{ii} + \sum_{i \neq j = 1}^{3} \mathrm{e}^{\gamma_{ij} u} (-1)^{\epsilon_i \epsilon_j} \boldsymbol{E}_{ij} \otimes \boldsymbol{E}_{ji}.$$

$\mathcal{L}(u)$ and $\mathcal{T}(u)$ satisfy the classical YBR

$$[\mathcal{L}_{01}(u_1), \mathcal{L}_{02}(u_2)] = [\mathcal{L}_{01}(u_1) + \mathcal{L}_{02}(u_2), r_{12}(u_1 - u_2)]$$
$$[\mathcal{T}_{01}(u_1), \mathcal{T}_{02}(u_2)] = [\mathcal{T}_{01}(u_1) + \mathcal{T}_{02}(u_2), r_{12}(u_1 - u_2)] \tag{16}$$

where $\mathcal{L}_{01}(u) = \mathcal{L}_n(u) \otimes I$, $\mathcal{L}_{02}(u) = I \otimes \mathcal{L}_n(u)$, $\mathcal{T}_{01}(u) = \mathcal{T}(u) \otimes I$ and $\mathcal{T}_{02}(u) = I \otimes \mathcal{T}(u)$.

Similarly, we expand the transfer matrix to order η^2

$$t(u) = -1 + \eta \,\mathrm{str}\mathcal{T}(u) + \eta^2 \mathrm{str}\mathcal{T}_2(u) + \mathrm{o}(\eta^3) \tag{17}$$

and the quantum determinant (7) would be

$$\Delta(u) = -1 + 3\eta \,\mathrm{str}\mathcal{T}(u) + \eta^2[\tfrac{3}{2}\mathrm{str}\mathcal{T}^2(u) - \tfrac{3}{2}(\mathrm{str}\mathcal{T}(u))^2 + 3\mathrm{str}\mathcal{T}^{(2)}(u) - 3\mathrm{str}\mathcal{T}^{(1)}(u)] + \mathrm{o}(\eta^3) \tag{18}$$

where $\mathcal{T}^{(1)}(u)$ means the first-order deviation of $\mathcal{T}(u)$ and the super-trace of $\mathcal{T}(u)$ and $\mathcal{T}^{(1)}(u)$ is

$$\mathrm{str}\mathcal{T}(u) = \sum_{n=1}^{N} \frac{\cosh(u - \delta_n)}{\sinh(u - \delta_n)} \qquad \mathrm{str}\mathcal{T}^{(1)}(u) = \sum_{n=1}^{N} \frac{1}{\sinh^2(u - \delta_n)}. \tag{19}$$

Now, we are ready to define the generating function of the Gaudin model at hand. The generalized $SU_q(1|2)$ Gaudin model is defined by

$$\hat{\tau}(u) = \frac{3}{2}\mathrm{str}\mathcal{T}^2(u) = \frac{3}{2} \sum_{n=1}^{N} \frac{2 + \cosh^2(u - \delta_n)}{\sinh^2(u - \delta_n)} + \sum_{n=1}^{N} \frac{\hat{H}_n}{\tanh(u - \delta_n)}. \tag{20}$$

In the second identity, we have used the definition of $\mathcal{T}(u)$ in (15), and the N independent commutative Hamiltonians are

$$\hat{H}_n = \sum_{m=1, m \neq n}^{N} \sum_{\alpha_1, \alpha_2 = 1}^{3} \frac{2(-1)^{\alpha_2} E_n^{\alpha_1 \alpha_2} E_m^{\alpha_2 \alpha_1}}{\tanh(\delta_n - \delta_m)} \qquad \sum_{n=1}^{N} \hat{H}_n = 0. \tag{21}$$

Based on the identities (17) and (18), we can express the generating function $\hat{\tau}(u)$ in terms of $t(u)$ and $\Delta(u)$

$$\tfrac{3}{2}\mathrm{str}\mathcal{T}^2(u) = (\eta)^{-2}\{\Delta(u) - 3t(u) - 2\} + (\eta)^2\left\{\tfrac{3}{2}(\mathrm{str}\mathcal{T}(u))^2 + 3\mathrm{str}\mathcal{T}^{(1)}(u)\right\}. \tag{22}$$

This means that $\hat{\tau}(u)$ constitutes a commutative family, i.e. $[\hat{\tau}(u), \hat{\tau}(v)] = 0$. There are two ways to show this. One is to use the classical YBR (16). Another way is based on the fact that the quantum determinant is the centre of the algebra and $[t(u), t(v)] = 0$. Thus, the Gaudin model defined above is integrable. Using the Jordan–Wigner transformation, the Hamiltonians can be expressed in terms of the fermionic creation and annihilation operators

$$\hat{H}_n = \sum_{m=1,m\neq n}^{N} \sum_{\sigma=\uparrow,\downarrow} \frac{3}{\tanh(\delta_n - \delta_m)} \{ (C^\dagger_{n,\sigma} C_{m,\sigma} + C^\dagger_{m,\sigma} C_{n,\sigma}) + 1 - n_n - n_m \\ -2S^+_n S^-_m - 2S^-_n S^+_m - 2S^z_n S^z_m + \tfrac{1}{2} n_n n_m \} \tag{23}$$

where

$$\begin{aligned} C^\dagger_{n,\uparrow} &= E^{13}_n \mathrm{e}^{-\mathrm{i}\pi \sum_{l=1}^{n-1} n_{n\uparrow}} \qquad C_{n,\uparrow} = \mathrm{e}^{\mathrm{i}\pi \sum_{l=1}^{n-1} n_{n\uparrow}} E^{31}_n \\ C^\dagger_{n,\downarrow} &= E^{23}_n \mathrm{e}^{-\mathrm{i}\pi \sum_{l=1}^{n-1} n_{n\downarrow}} \mathrm{e}^{-\mathrm{i}\pi \sum_{k=1}^{L} n_{k\downarrow}} \qquad C_{n,\downarrow} = \mathrm{e}^{\mathrm{i}\pi \sum_{k=1}^{L} n_{k\downarrow}} \mathrm{e}^{\mathrm{i}\pi \sum_{l=1}^{n-1} n_{n\downarrow}} E^{32}_n \\ S^+_n &= \frac{1}{\sqrt{2}} C^\dagger_{n,\uparrow} C_{n,\downarrow} \qquad S^-_n = \frac{1}{\sqrt{2}} C^\dagger_{n,\downarrow} C_{n,\uparrow} \qquad S^z_j = \tfrac{1}{2}(C^\dagger_{n,\uparrow} C_{n,\uparrow} - C^\dagger_{n,\downarrow} C_{n,\downarrow}) \end{aligned} \tag{24}$$

and L is the length of the lattice.

We assume the eigenstate of $\hat{\tau}(u)$ to be of the form

$$\Phi^3 = \sum_{\alpha_1\alpha_2\cdots\alpha_m}^{2} \mathcal{C}_{\alpha_1}(\lambda_1)\mathcal{C}_{\alpha_2}(\lambda_2)\cdots\mathcal{C}_{\alpha_m}(\lambda_m)|\Omega\rangle F^{\alpha_m\cdots\alpha_2\alpha_1} \tag{25}$$

where the number of α_j taking the value of unity is m_1 and the coefficients $F^{\alpha_m\cdots\alpha_2\alpha_1}$ are related to the following states:

$$|\lambda^{(1)}_1, \lambda^{(1)}_2, \ldots, \lambda^{(1)}_{m_1}\rangle = \mathcal{C}^{(1)}(\lambda^{(1)}_1)\mathcal{C}^{(1)}(\lambda^{(1)}_2)\cdots\mathcal{C}^{(1)}(\lambda^{(1)}_{m_1})|\omega\rangle. \tag{26}$$

The state $|\lambda^{(1)}_1, \lambda^{(1)}_2, \ldots, \lambda^{(1)}_{m_1}\rangle$ 'lives' on a lattice of m sites and is thus an element of a direct product over m Hilbert spaces. In components it reads $|\lambda^{(1)}_1, \lambda^{(1)}_2, \ldots, \lambda^{(1)}_{m_1}\rangle_{\alpha_m\cdots\alpha_2\alpha_1}$ and can be directly identified with $F^{\alpha_m\cdots\alpha_2\alpha_1}$. The operators $\mathcal{C}_{\alpha_l}(\lambda_l)$ and $\mathcal{C}^{(1)}(\lambda^1)$ are

$$\mathcal{C}_{\alpha_l}(\lambda_l) = \sum_{n=1}^{N} \frac{\mathrm{e}^{-(\lambda_l-\delta_n)}}{\sinh(\lambda_l - \delta_n)} E^{\alpha_l 3}_n \qquad \mathcal{C}^{(1)}(\lambda^{(1)}_j) = \sum_{n=1}^{m} \frac{\mathrm{e}^{-(\lambda^{(1)}_j-\lambda_n)}}{\sinh(\lambda^{(1)}_j - \lambda_n)} e^{12}_n \tag{27}$$

where e^{12}_n are the quantum operators of the auxiliary L-matrix in the second row and first column. We also have $|\omega\rangle_n = (0,1)^t_n$ and $|\omega\rangle = |\omega\rangle_1 \otimes |\omega\rangle_2 \cdots |\omega\rangle_m$. It is worth pointing out that the operators $\mathcal{C}_{\alpha_l}(\lambda)$ are the entries of the monodromy matrix $\mathcal{T}(\lambda)$ in the third row and α_lth column, while the operator $\mathcal{C}^{(1)}(u)$ is the entry of an auxiliary monodromy matrix (six-vertex) in the second row and first column. This auxiliary monodromy matrix acts on the α_l configuration (spin configuration) in the first step of the Bethe ansatz method. The quantities m and m_1 can also be identified as the total number of electrons and the number of spin-up electrons respectively, i.e. $m = N_e = N_\uparrow + N_\downarrow$ and $m_1 = N_\uparrow$. Thus, by using the graded QISM, the eigenstate of the generalized $SU_q(1|2)$ Gaudin model is

$$|\lambda_1, \lambda_2, \ldots, \lambda_m|F\rangle = \prod_{j=1}^{m}\sum_{n=1}^{N} \frac{\mathrm{e}^{-(\lambda_l-\delta_n)}}{\sinh(\lambda_j - \delta_m)} E^{\alpha_j 3}_n |\Omega\rangle \prod_{k=1}^{m_1}\sum_{n=1}^{m} \frac{\mathrm{e}^{-(\lambda^{(1)}_k-\lambda_n)}}{\sinh(\lambda^{(1)}_k - \lambda_n)} e^{12}_n |\omega\rangle \tag{28}$$

if the following Bethe ansatz equations are valid:

$$\sum_{n=1}^{N} \coth(\lambda_k - \delta_n) = \sum_{j=1}^{m_1} \coth(\lambda_k - \lambda^{(1)}_j) \tag{29}$$

$$\sum_{i=1}^{m} \coth(\lambda_i - \lambda^{(1)}_l) = 2 \sum_{j=1,j\neq l}^{m_1} \coth(\lambda^{(1)}_j - \lambda^{(1)}_l). \tag{30}$$

The transfer matrix $t(u)$ acting on the eigenstate gives the eigenvalue

$$\begin{aligned}\Lambda_{t(u)} = -1 + \eta \sum_{n=1}^{N} \coth(u-\delta_n) + \eta^2 \Bigg\{ \frac{N}{2} - m_1 + \sum_{n=1}^{N-1} \sum_{l=n+1}^{N} \coth(u-\delta_n)\coth(u-\delta_l) \\ -2 \sum_{k=1}^{m_1-1} \sum_{p=k+1}^{m_1} \coth(\lambda_k^{(1)} - u)\coth(\lambda_p^{(1)} - u) \\ + \sum_{n=1}^{N} \coth(u-\delta_n) \sum_{j=1}^{m} \coth(\lambda_j - u) \\ - \sum_{k=1}^{m_1} \coth(u - \lambda_k^{(1)}) \sum_{j=1}^{m} \coth(\lambda_j - u) \Bigg\} + \mathrm{o}(\eta^3).\end{aligned} \quad (31)$$

Thanks to equations (14), (22) and (31), the eigenvalue of the generating function of the generalized $SU_q(1|2)$ Gaudin model is

$$\begin{aligned}\Lambda_{\hat{\tau}(u)} = 3m_1 + 6 \sum_{k=1}^{m_1-1} \sum_{p=k+1}^{m_1} \coth(\lambda_k^{(1)} - u)\coth(\lambda_p^{(1)} - u) \\ -3 \sum_{n=1}^{N} \coth(u-\delta_n) \sum_{j=1}^{m} \coth(\lambda_j - u) \\ +3 \sum_{k=1}^{m_1} \coth(u - \lambda_k^{(1)}) \sum_{j=1}^{m} \coth(\lambda_j - u) \\ + \frac{3}{2} \left(\sum_{n=1}^{N} \coth(u-\delta_n) \right)^2 + 3 \sum_{n=1}^{N} \frac{1}{\sinh^2(u-\delta_n)}.\end{aligned} \quad (32)$$

Considering the residues of $\Lambda_{\hat{\tau}(u)}$ at $u = \delta_n$ and using the Bethe ansatz equations (29) and (30), we finally obtain the eigenvalues of the Hamiltonians $\hat{H}_n$:

$$\Lambda_{\hat{H}_n} = \sum_{m=1, m\neq n}^{N} 3\coth(\delta_n - \delta_m) - \sum_{j=1}^{m} 3\coth(\lambda_j - \delta_n) \qquad n = 1, 2, \ldots, N. \quad (33)$$

4. Discussion

We construct the Hamiltonians of the generalized $SU_q(1|2)$ Gaudin model based on the graded quantum determinant in the graded case. Meanwhile we give the eigenstate and the eigenvalues of the Hamiltonians and the generating functions of the generalized $SU_q(1|2)$ Gaudin model.

In this paper, all discussions are based on periodic boundary conditions. It is also interesting to study the boundary behaviour of the quantum integrable model in many kinds of boundary condition, so one can study the Gaudin model in open boundary conditions. In that case, one may not use the quantum determinant. We will give these results in future papers.

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References

[1] Gaudin M 1972 Modèles exacts en mécanique statistique: la méthode de Bethe et ses généralizations *Note CEA* **1559**(1)
Gaudin M 1973 Modèles exacts en mécanique statistique: la méthode de Bethe et ses généralizations *Note CEA* **1559**(2)
[2] Gaudin M 1976 *J. Physique* **37** 1087
[3] Gaudin M 1983 *La Fonction D'onde de Bethe* (Paris: Masson)
[4] Babujian H M and Flume R 1994 *Mod. Phys. Lett.* A **9** 2029
[5] Feigin B, Frenkel E and Reshetikhin N 1994 *Commun. Math. Phys.* **166** 27
[6] Faddeev L D 1982 Integrable model in (1 + 1)-dimensional quantum field theory *Recent Advances in Field Theory and Statistical Mechanics* ed J B Zuber and R Stora (Amsterdam: Elsevier Science) pp 561–608
[7] Kulish P P and Sklyanin E K 1982 The quantum spectral transform method. Recent developments *Lett. Notes Phys.* **151** 61
[8] Sklyanin E K 1989 *J. Sov. Math.* **47** 2473
[9] Sklyanin E K and Takebe T 1996 *Phys. Lett.* A **219** 217
[10] Takebe T 1992 *J. Phys. A: Math. Gen.* **25** 1071
[11] Takebe T 1995 *J. Phys. A: Math. Gen.* **28** 6675
[12] Anderson P W 1988 *Frontiers and Borderlines in Many Particle Physics* ed R A Broglia and J R Schrieffer (Amsterdam: North-Holland)
[13] Fukuyama H, Maekawa S and Malozemoff A P 1989 *Strongly Correlation and Super-Conductivity* (Berlin: Springer)
[14] Anderson P W 1987 *Science* **235** 1196
Anderson P W 1990 *Phys. Rev. Lett.* **65** 2306
[15] Zhang F C and Rice T M 1988 *Phys. Rev.* B **37** 3759
[16] Fei S M and Yue R H 1994 *J. Phys. A: Math. Gen.* **27** 3715
[17] Essler F H L and Korepin V E 1992 *Phys. Rev.* B **46** 9147
[18] Takhtajan L A and Faddeev L D 1979 *Russ. Math. Surv.* **34** 11
[19] Korepin V E, Izergin G and Bogoliubov N M 1992 *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge: Cambridge University Press)
[20] Yue R H and Qiu Z M 1996 *Commun. Theor. Phys.* **26** 389
[21] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[22] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[23] Perk J H and Schultz C L 1981 *Phys. Lett.* A **84** 407
[24] Hikami K, Kulish P P and Wadati M 1992 *J. Phys. Soc. Japan* **61** 3071